

# Quantum supersymmetric Toda-mKdV hierarchies

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## Abstract

In this paper we generalize the quantization procedure of Toda-mKdV hierarchies to the case of arbitrary affine (super)algebras. The quantum analogue of the monodromy matrix, related to the universal R-matrix with the lower Borel subalgebra represented by the corresponding vertex operators is introduced. The auxiliary L-operators satisfying RTT-relation are constructed and the quantum integrability condition is obtained. General approach is illustrated by means of two important examples.

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## 1 Introduction

During more than quarter of a century both classical and quantum affine Toda field theories and related generalized (m)KdV ((modified) Korteweg-de Vries) hierarchies were extensively studied (see e.g. [1]-[6]). These theories are integrable and have the L-A pair (or zero curvature) formulation. The most famous are sine-Gordon and  $A_2^{(2)}$  models (see e.g. [7], [8]) and the associated KdV and reduced Boussinesq hierarchies. These theories allow the supersymmetric and fermionic generalizations [9]-[11] both called “super” because their underlying algebraic structures are affine Lie superalgebras.

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The super-Toda field theory appears to be supersymmetric if and only if the associated superalgebra possesses the purely fermionic system of simple roots. One should note, however, that usually the superalgebras allow a few simple root systems [12], they correspond to different Toda theories and only purely fermionic system corresponds to the supersymmetric one.

The supersymmetric version of the Drinfeld-Sokolov reduction applied to the matrix L-operator of the Toda-mKdV theories gives the generators of the related super W-algebra with the commutation relations provided by the associated Hamiltonian structure (see e.g. [13], [14]).

In this paper we consider these theories from a point of view of quantum inverse scattering method (QISM) (see [15], [16]). There are two approaches of applying QISM to Toda-mKdV type models. The first one is more traditional and based on the quantization of the corresponding lattice systems (see e.g. [17],[18]) and the second one is based on the quantization in terms of continuous free field theory and was introduced in [3], [4].

Here we use the second approach, generalizing our results obtained in [20]-[22] for the affine superalgebras of rank 2 to the case of general affine superalgebra.

We build the quantum generalization of the monodromy matrix and prove that the related auxiliary  $\mathbf{L}$ -operators (which are equal to the monodromy matrix multiplied by the exponential of the elements from the Cartan subalgebra) satisfy the RTT-relation [15], [16], while the quantum counterpart of the monodromy matrix itself is shown to satisfy the specialization of the reflection equation (see e.g. [19]). This provides the quantum integrability relation for the supertraces of the monodromy matrix taken in different representations (transfer matrices). Moreover, it is proven that the auxiliary  $\mathbf{L}$ -operators are related with the universal R-matrix associated with the underlying quantum affine superalgebra, with the lower Borel subalgebra represented by the vertex operators from the corresponding Toda field theory. Using this relation in the case when the simple root system is purely fermionic, it is demonstrated that the transfer matrix is invariant under the supersymmetry transformation as it was shown on particular classical examples of Toda field theories [10], [11].

In the last two sections the above constructions are illustrated by means of two important examples of integrable hierarchies: quantum super-KdV [20] and SUSY N=1 KdV [21]. These hierarchies generate two integrable structures of the superconformal field theory, the second one is invariant under the SUSY transformation while the first one is not.

## 2 Bosonic Toda-mKdV hierarchies

Each mKdV hierarchy and related Toda field theory associated with affine Lie algebra are generated by the following L-operator [1]:

$$\mathcal{L} = \partial_u - \partial_u \phi^i(u) H^i - \left( \sum_{i=0}^r e_{\alpha_i} \right), \quad (1)$$

where  $u$  lies on a cylinder of circumference  $2\pi$ ,  $\phi^i$  are the scalar fields with the Poisson brackets:

$$\{\partial_u \phi^i(u), \partial_v \phi^j(v)\} = -\delta^{ij} \delta'(u-v) \quad (2)$$

with quasiperiodic boundary condition:

$$\phi^i(u+2\pi) = \phi^i(u) + 2\pi i p^i. \quad (3)$$

$e_{\alpha_i}$  are the Chevalley generators of the underlying affine Lie algebra and  $H^i$  ( $i = 1, \dots, r$ ) form a basis in the Cartan subalgebra of the corresponding simple Lie algebra:

$$[H^i, e_{\alpha_k}] = \alpha_k^i e_{\alpha_k}, \quad [e_{\alpha_k}, e_{-\alpha_l}] = \delta_{kl} h_{\alpha_k}, \quad ad_{e_{\pm\alpha_k}}^{1-a_{kj}} e_{\pm\alpha_j} = 0, \quad (4)$$

where  $a_{kj}$  is a Cartan matrix and  $h_{\alpha_k} \equiv (\alpha_k, H) = \alpha_k^i H^i$ . In our case this algebra is considered in evaluation representations, when  $e_{\alpha_0} = \lambda e_{-\theta}$  (this corresponds to the case of untwisted affine Lie algebra, the twisted case is more complicated),  $\theta$  is the highest root of the related simple Lie algebra. The classical monodromy matrix for the linear problem associated with the L-operator (1) can be expressed in the following way [3]:

$$\pi_s(\lambda)(\mathbf{M}) \equiv \mathbf{M}_s(\lambda) = e^{2\pi i p^k H^k} \text{Pexp} \int_0^{2\pi} du \left( \sum_{i=0}^r e^{-(\alpha_i, \phi(u))} e_{\alpha_i} \right), \quad (5)$$

where  $\pi_s$  is some evaluation representation of the corresponding affine Lie algebra. Defining the auxiliary  $\mathbf{L}$ -matrix:

$$\mathbf{L}(\lambda) = e^{-\pi i p^k H^k} \mathbf{M}(\lambda) \quad (6)$$

one can find that the quadratic Poisson bracket relation is valid:

$$\{\mathbf{L}(\lambda) \otimes, \mathbf{L}(\mu)\} = [\mathbf{r}(\lambda \mu^{-1}), \mathbf{L}(\lambda) \otimes \mathbf{L}(\mu)], \quad (7)$$

where  $\mathbf{r}(\lambda)$  is the trigonometric r-matrix [16], [23] related with the corresponding simple Lie algebra. The traces of the monodromy matrices in different evaluation representations  $\pi_s$ :

$$\mathbf{t}_s(\lambda) = \pi_s(\mathbf{M}(\lambda)) \quad (8)$$

are in involution under the Poisson brackets:

$$\{\mathbf{t}_s(\lambda), \mathbf{t}_{s'}(\mu)\} = 0. \quad (9)$$

The quantization means that we move from the quadratic Poisson bracket relation (7) to the RTT-relation with the underlying affine Lie algebra deformed to the quantum affine algebra (see e.g. [15], [16] and below). In this section we give the generalization of the constructions appeared in [4], [5].

First, let's quantize the scalar fields  $\phi^i$ :

$$\phi^k(u) = iQ^k + iP^k u + \sum_n \frac{a_{-n}^k}{n} e^{inu}, \quad (10)$$

$$[Q^k, P^j] = \frac{i}{2} \beta^2 \delta^{kj}, \quad [a_n^k, a_m^j] = \frac{\beta^2}{2} n \delta^{kj} \delta_{n+m,0},$$

and define the vertex operators

$$\begin{aligned} \tilde{V}_{\alpha_k}(u) &:= e^{-(\alpha_k, \phi(u))} := \exp \left( \sum_{n=1}^{\infty} \frac{(-\alpha_k, a_{-n})}{n} e^{inu} \right) \\ &\exp \left( -i((\alpha_k, Q) + (\alpha_k, P)u) \right) \exp \left( \sum_{n=1}^{\infty} \frac{(\alpha_k, a_n)}{n} e^{-inu} \right). \end{aligned}$$

Then one can define the quantum generalizations of auxiliary  $\mathbf{L}$ -operators [3]:

$$\mathbf{L}^{(q)}(\lambda) = e^{\pi i P^k H^k} P \exp \int_0^{2\pi} du \left( \sum_{i=0}^r : e^{-(\alpha_i, \phi(u))} : e_{\alpha_i} \right), \quad (11)$$

where now  $e_{\alpha_i}$ ,  $H^k$  are the generators of the corresponding quantum affine algebra:

$$[H^i, e_{\alpha_k}] = \alpha_k^i e_{\alpha_k}, \quad [e_{\alpha_k}, e_{-\alpha_l}] = \delta_{kl} [h_{\alpha_k}]_q, \quad ad_{e_{\pm \alpha_k}}^{(q)^{1-a_{kj}}} e_{\pm \alpha_j} = 0, \quad (12)$$

where  $q = e^{i\pi \frac{\beta^2}{2}}$ ,  $[a]_q = (q^a - q^{-a})/(q - q^{-1})$  and  $ad_{e_{\alpha}}^{(q)} e_{\beta} = e_{\alpha} e_{\beta} - q^{(\alpha, \beta)} e_{\beta} e_{\alpha}$ . The object (11) is defined in the interval  $0 < \beta^2 < 2b^{-1}$ ,  $b = \max(|b_{ij}|)$  ( $i \neq j$ ), where  $b_{ij}$  is a symmetrized Cartan matrix, but can be analytically continued to a wider region. The quantum monodromy matrix is defined using the relation (6):

$$\mathbf{M}^{(q)}(\lambda) = e^{\pi i P^k H^k} \mathbf{L}^{(q)}(\lambda). \quad (13)$$

It can be shown that the  $\mathbf{L}^{(q)}$  operator satisfies the mentioned RTT-relation in the two ways [4], [5]. The first way is to consider the following product:

$$(\mathbf{L}^{(q)}(\lambda) \otimes I)(I \otimes \mathbf{L}^{(q)}(\mu)). \quad (14)$$

Then, moving all the Cartan multipliers to the left we find the following:

$$e^{i\pi P^i \Delta(H^i)} \text{Pexp} \int_0^{2\pi} du \tilde{K}_1(u) \text{Pexp} \int_0^{2\pi} du K_2(u), \quad (15)$$

$$K_1(u) = \sum_{j=0}^r : e^{-(\alpha_j, \phi(u))} : e_{\alpha_j} \otimes q^{h_{\alpha_j}}, \quad K_2(u) = \sum_{j=0}^r 1 \otimes : e^{-(\alpha_j, \phi(u))} : e_{\alpha_j},$$

where  $\Delta(H^i) = H^i \otimes I + I \otimes H^i$ . The commutation relations between vertex operators on a circle:

$$\tilde{V}_{\alpha_k}(u) \tilde{V}_{\alpha_j}(u') = q^{b_{kj}} \tilde{V}_{\alpha_j}(u') \tilde{V}_{\alpha_k}(u), \quad u > u', \quad (16)$$

lead to

$$[\tilde{K}_1(u), K_2(u')] = 0, \quad u < u'. \quad (17)$$

Due to this property one can unify two P-exponents into the single one which is equal to

$$\pi_s(\lambda) \otimes \pi_{s'}(\mu) \Delta(\mathbf{L}^{(q)}), \quad (18)$$

where  $\pi_s$  and  $\pi_{s'}$  are some evaluation representations, and the coproduct  $\Delta$  of the quantum affine (super)algebra is defined by:

$$\begin{aligned} \Delta(H^i) &= H^i \otimes I + I \otimes H^i, \quad \Delta(e_{\alpha_j}) = e_{\alpha_j} \otimes q^{h_{\alpha_j}} + 1 \otimes e_{\alpha_j}, \\ \Delta(e_{-\alpha_j}) &= e_{-\alpha_j} \otimes 1 + q^{-h_{\alpha_j}} \otimes e_{-\alpha_j}. \end{aligned} \quad (19)$$

Next, considering opposite product of the  $\mathbf{L}$ -operators, one finds that it coincide with opposite coproduct of the  $\mathbf{L}$ -operators:

$$(I \otimes \mathbf{L}^{(\mathfrak{q})}(\mu))(\mathbf{L}^{(\mathfrak{q})}(\lambda) \otimes I) = \pi_s(\lambda) \otimes \pi_s(\mu) \Delta^{op}(\mathbf{L}^{(\mathfrak{q})}), \quad (20)$$

where  $\Delta^{op} = \tau \Delta$  and the map  $\tau$  is defined as follows:  $\tau(a \otimes b) = b \otimes a$ . Using the property of the universal R-matrix, namely  $\mathbf{R} \Delta = \Delta^{op} \mathbf{R}$  [28], we arrive to the RTT-relation:

$$\begin{aligned} \mathbf{R}(\lambda\mu^{-1})(\mathbf{L}^{(q)}(\lambda) \otimes I)(I \otimes \mathbf{L}^{(q)}(\mu)) = \\ (I \otimes \mathbf{L}^{(q)}(\mu))(\mathbf{L}^{(q)}(\lambda) \otimes I)\mathbf{R}(\lambda\mu^{-1}). \end{aligned} \quad (21)$$

Remembering the expression for the monodromy matrix (13) one obtains that the RTT-relation is no longer valid for the monodromy matrices, however, it is easy to see that multiplying both RHS and LHS of (21) by  $e^{i\pi\Delta H^k P^k}$  one obtains:

$$\mathbf{R}_{12}(\lambda\mu^{-1})\tilde{\mathbf{M}}_1^{(q)}(\lambda)\mathbf{M}_2^{(q)}(\mu) = \tilde{\mathbf{M}}_2^{(q)}(\mu)\mathbf{M}_1^{(q)}(\lambda)\mathbf{R}_{12}(\lambda\mu^{-1}), \quad (22)$$

where we have denoted  $\tilde{\mathbf{M}}_1^{(q)}(\lambda)$  the monodromy matrix with  $e_{\alpha_i} \otimes 1$  replaced by  $e_{\alpha_i} \otimes q^{-h_{\alpha_i}}$  and  $\tilde{\mathbf{M}}_2^{(q)}(\lambda)$  the monodromy matrix with  $1 \otimes e_{\alpha_i}$  replaced by  $q^{-h_{\alpha_i}} \otimes e_{\alpha_i}$ . Taking the trace, the above additional Cartan factors in  $\tilde{\mathbf{M}}^{(q)}$  cancel and we obtain the quantum integrability condition for their traces (transfer-matrices):

$$[\mathbf{t}_s(\lambda), \mathbf{t}_{s'}(\mu)] = 0. \quad (23)$$

Another more universal way to obtain the RTT-relation is the correspondence between the reduced universal R-matrix (see below) and the P-exponential form of the auxiliary  $\mathbf{L}^{(q)}$  operator. That is, let's consider integrals of vertex operators:

$$V_{\alpha_k}(u_2, u_1) = \frac{1}{q - q^{-1}} \int_{u_1}^{u_2} du \tilde{V}_{\alpha_k}(u). \quad (24)$$

Via the contour technique [2] one can show that these objects satisfy the quantum Serre relations of the lower Borel subalgebra of the associated quantum affine algebra with simple roots  $\alpha_k$ . Using the structure of the reduced universal R-matrix (see e.g. [27] and Appendix) we can write  $\bar{R} = K^{-1}\mathbf{R} = \bar{R}(\bar{e}_{\alpha_i}, \bar{e}_{-\alpha_i})$ , where

$$\bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1, \quad \bar{e}_{-\alpha_i} = 1 \otimes e_{-\alpha_i}, \quad (25)$$

$\mathbf{R}$  is a universal R-matrix and  $K$  depends on the elements from Cartan subalgebra, because  $\bar{R}$  is represented as a power series of these elements. Then, following [5] and using the fundamental feature of the universal R-matrix:

$$(I \otimes \Delta)\mathbf{R} = \mathbf{R}^{13}\mathbf{R}^{12}, \quad (26)$$

one can show that the reduced R-matrix has the following property:

$$\bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i} + e''_{-\alpha_i}) = \bar{R}(\bar{e}_{\alpha_i}, e'_{-\alpha_i})\bar{R}(\bar{e}_{\alpha_i}, e''_{-\alpha_i}), \quad (27)$$

where

$$\begin{aligned} e'_{-\alpha_i} + e''_{-\alpha_i} &= (I \otimes \Delta)(1 \otimes e_{-\alpha_i}), \\ e'_{-\alpha_i} &= 1 \otimes q^{-h_{\alpha_i}} \otimes e_{-\alpha_i}, \quad e''_{-\alpha_i} = 1 \otimes e_{-\alpha_i} \otimes 1, \quad \bar{e}_{\alpha_i} = e_{\alpha_i} \otimes 1 \otimes 1. \end{aligned} \quad (28)$$

Their commutation relations are

$$\begin{aligned} e'_{-\alpha_i} \bar{e}_{\alpha_j} &= \bar{e}_{\alpha_j} e'_{-\alpha_i}, \quad e''_{-\alpha_i} \bar{e}_{\alpha_j} = \bar{e}_{\alpha_j} e''_{-\alpha_i}, \\ e'_{-\alpha_i} e''_{-\alpha_j} &= q^{b_{ij}} e''_{-\alpha_j} e'_{-\alpha_i}. \end{aligned} \quad (29)$$

Now, denoting by  $\bar{\mathbf{L}}^{(q)}(u_2, u_1)$  the reduced R-matrix with  $e_{-\alpha_i}$  represented by  $V_{\alpha_i}(u_2, u_1)$  and using the above property of  $\bar{R}$  with  $e'_{-\alpha_i}$  replaced by appropriate vertex operators we find:

$$\bar{\mathbf{L}}^{(q)}(u_3, u_1) = \bar{\mathbf{L}}^{(q)}(u_3, u_2) \bar{\mathbf{L}}^{(q)}(u_2, u_1), \quad u_3 \geq u_2 \geq u_1. \quad (30)$$

Hence,  $\bar{\mathbf{L}}^{(q)}$  has the property of P-exponent. When the interval  $\delta = [u_2, u_1]$  is small enough one can show that

$$\bar{\mathbf{L}}^{(q)}(u_2, u_1) = 1 + \int_{u_1}^{u_2} du \left( \sum_{i=0}^r : e^{-(\alpha_i, \phi(u))} : e_{\alpha_i} \right) + O(\delta^2). \quad (31)$$

That is, we obtain that

$$\bar{\mathbf{L}}^{(q)}(u_2, u_1) = P \exp \int_{u_1}^{u_2} du \left( \sum_{i=0}^r : e^{-(\alpha_i, \phi(u))} : e_{\alpha_i} \right) \quad (32)$$

and  $\mathbf{L}^{(q)} = e^{i\pi H^i P^i} \bar{\mathbf{L}}^{(q)}(2\pi, 0)$  satisfies RTT relation by construction.

### 3 Quantum P-exponential and Toda-mKdV hierarchies based on superalgebras

Now let's generalize the above results to the case when the underlying algebraic structures and integrable hierarchies are related to the affine Lie superalgebra. In the previous part we have moved from classical theory to the quantum one, here we will go in opposite direction, moving from the quantum version of the monodromy matrix and related auxiliary  $\mathbf{L}$ -operators, satisfying RTT-relation to their classical counterparts.

First, let's introduce two types of vertex operators, bosonic and fermionic ones:

$$W_{\alpha_i}^F(u) \equiv \int d\theta : e^{-(\alpha_i, \Phi(u, \theta))} := \frac{i}{\sqrt{2}} (\alpha_i, \xi(u)) : e^{-(\alpha_i, \phi(u))} : \quad (33)$$

$$W_{\alpha_i}^B(u) \equiv \int d\theta \theta : e^{-(\alpha_i, \Phi(u, \theta))} := e^{-(\alpha_i, \phi(u))} :, \quad (34)$$

where  $\Phi^k$  are the superfields:  $\Phi^k(u, \theta) = \phi^k(u) - \frac{i}{\sqrt{2}} \theta \xi^k(u)$  and  $\theta$  is a Grassmann variable. Their commutation relations on a circle are:

$$W_{\alpha_i}^s(u) W_{\alpha_k}^{s'}(u') = (-1)^{p(s)p(s')} q^{b_{ik}} W_{\alpha_k}^{s'}(u') W_{\alpha_i}^s(u), \quad u > u', \quad (35)$$

where  $b_{kj}$  is the symmetrized Cartan matrix for the corresponding affine Lie superalgebra,  $s, s'$  are  $B, F$  and  $p(F) = 1, p(B) = 0$ .

The mode expansion for the bosonic fields is the same as in (10) and for fermionic fields  $\xi^k(u)$  is the following:

$$\xi^l(u) = i^{-1/2} \sum_n \xi_n^l e^{-inu}, \quad \{\xi_n^k, \xi_m^l\} = \beta^2 \delta^{kl} \delta_{n+m, 0}. \quad (36)$$

These fermion fields may satisfy two boundary conditions periodic and antiperiodic  $\xi^i(u + 2\pi) = \pm \xi^i(u)$  corresponding to the two sectors of (S)CFT – Ramond (R) and Neveu-Schwarz (NS) (the supersymmetry operator appears only when all fermions are in the R sector).

It can be shown that the integrals of the introduced vertex operators as in the bosonic case satisfy the Serre and “non Serre” relations (see e.g. [25]- [27]) for the lower Borel subalgebra:

$$ad_{e_{-\alpha_k}}^{(q)^{1-a_{kj}}} e_{-\alpha_j} = 0, \quad [[e_{\pm\alpha_r}, e_{\pm\alpha_s}]_q, [e_{\pm\alpha_r}, e_{\pm\alpha_p}]_q]_q = 0, \quad (37)$$

where the  $ad_{e_{-\alpha_k}}^{(q)}$  operator is defined in the following way:  $ad_{e_{-\alpha_k}}^{(q)} e_{\beta} = e_{\alpha} e_{\beta} - (-1)^{p(\alpha)p(\beta)} q^{(\alpha, \beta)} e_{\beta} e_{\alpha}$ , and  $e_{\pm\alpha_r}$  is a so-called “grey” root (for more details see Appendix) of the quantum affine superalgebra with the corresponding bosonic and fermionic roots  $\alpha_i$ .

Substituting them with the appropriate multiplier  $(q - q^{-1})^{-1}$  in the reduced R-matrix one can find (easily generalizing the results of section 2 to the case of superalgebra) that it satisfies the P-exponential multiplication property:

$$\begin{aligned} \bar{\mathbf{L}}^{(q)}(u_2, u_1) &= Pexp^{(q)} \int_{u_1}^{u_2} du \left( \sum_f W_{\alpha_f}^F(u) e_{\alpha_f} + \sum_b W_{\alpha_b}^B(u) e_{\alpha_b} \right) \\ \bar{\mathbf{L}}^{(q)}(u_3, u_1) &= \bar{\mathbf{L}}^{(q)}(u_3, u_2) \bar{\mathbf{L}}^{(q)}(u_2, u_1), \quad u_3 \geq u_2 \geq u_1, \end{aligned} \quad (38)$$



where indices  $f$  and  $b$  imply that we are summing over fermionic and bosonic simple roots. The letter  $q$  over the  $Pexp$  means that the object introduced above in some cases (more precisely when a number of fermionic roots is more than one) cannot be written as P-exponential for any value of the deformation parameter due to the singularities in the operator products generated by the fermion fields  $\xi^i$ . Thus we call this object quantum P-exponential.

Defining then  $\mathbf{L}^{(q)} \equiv e^{\pi i p^i H^i} \bar{\mathbf{L}}^{(q)}(2\pi, 0)$  we find (similarly to the purely bosonic case) that it satisfies the RTT relation (21) and defining the monodromy matrix  $\mathbf{M}^{(q)} \equiv e^{\pi i p^i H^i} \mathbf{L}^{(q)}$  we again arrive to the property (22) and obtain again the quantum integrability condition (23) for  $\mathbf{t}^{(q)} = str \mathbf{M}^{(q)}$ . We mention here that the relation (22) can be rewritten in a more universal way, as a specialization of the reflection equation [19]:

$$\tilde{\mathbf{R}}_{12}(\lambda\mu^{-1})\mathbf{M}_1^{(q)}(\lambda)F_{12}^{-1}\mathbf{M}_2^{(q)}(\mu) = \mathbf{M}_2^{(q)}(\mu)F_{12}^{-1}\mathbf{M}_1^{(q)}(\lambda)\mathbf{R}_{12}(\lambda\mu^{-1}), \quad (39)$$

where  $F = K^{-1}$  the Cartan's factor from the universal R-matrix (see Appendix), and  $\tilde{\mathbf{R}}_{12}(\lambda\mu^{-1}) = F_{12}^{-1}\mathbf{R}_{12}(\lambda\mu^{-1})F_{12}$ .

Now let's analyse the classical limit of the defined objects. We will use the P-exponential property of  $\bar{\mathbf{L}}^{(q)}(2\pi, 0)$ . Let's decompose  $\bar{\mathbf{L}}^{(q)}(2\pi, 0)$  in the following way:

$$\bar{\mathbf{L}}^{(q)}(2\pi, 0) = \lim_{N \rightarrow \infty} \prod_{m=1}^N \bar{\mathbf{L}}^{(q)}(x_m, x_{m-1}), \quad (40)$$

where we divided the interval  $[0, 2\pi]$  into infinitesimal intervals  $[x_m, x_{m+1}]$  with  $x_{m+1} - x_m = \epsilon = 2\pi/N$ . Let's find the terms that can give contribution of the first order in  $\epsilon$  in  $\bar{\mathbf{L}}^{(q)}(x_m, x_{m-1})$ . In this analysis one needs the operator product expansion of fermion fields and vertex operators:

$$\begin{aligned} \xi^k(u)\xi^l(u') &= -\frac{i\beta^2\delta^{kl}}{(iu - iu')} + \sum_{p=0}^{\infty} c_p^{kl}(u)(iu - iu')^p, \\ :e^{-(\alpha_k, \phi(u))}::e^{-(\alpha_l, \phi(u'))} &:= \\ (iu - iu')^{\frac{(\alpha_k, \alpha_l)\beta^2}{2}} &(:e^{-(\alpha_k + \alpha_l, \phi(u))}: + \sum_{p=1}^{\infty} d_p^{kl}(u)(iu - iu')^p), \end{aligned} \quad (41)$$

where  $c_p^{kl}(u)$  and  $d_p^{kl}(u)$  are operator-valued functions of  $u$ . Now one can see that only two types of terms can give the contribution of the order  $\epsilon$  in  $\bar{\mathbf{L}}^{(q)}(x_{m-1}, x_m)$  when  $q \rightarrow 1$ . The first type consists of operators of the first order in  $W_{\alpha_i}$  and the second type is formed by the operators, quadratic in  $W_{\alpha_i}$ , which give contribution of the order  $\epsilon^{1 \pm \beta^2}$  by virtue of operator product expansion. Let's look on the terms of the second type in detail.

The terms of the second type appear from the quadratic products of vertex

operators arising from:

- i) the composite roots (more precisely  $q$ -commutators of two fermionic roots),
- ii) the quadratic terms of the  $q$ -exponentials which are present in the universal  $R$ -matrix.

At first we consider terms emerging from composite roots, which have the following form (see Appendix):

$$\frac{1}{a(\alpha_i + \alpha_j)(q - q^{-1})} [e_{\alpha_j}, e_{\alpha_i}]_{q^{-1}} \left( \int_{x_{m-1}}^{x_m} du_1 W_{\alpha_i}(u_1 - i0) \right. \\ \left. \int_{x_{m-1}}^{x_m} du_2 W_{\alpha_j}(u_2 + i0) + q^{b_{ij}} \int_0^{2\pi} du_2 W_{\alpha_j}(u_2 - i0) \int_0^{2\pi} du_1 W_{\alpha_i}(u_1 + i0) \right). \quad (42)$$

Using the fact that  $q = e^{i\pi \frac{\beta^2}{2}}$  and that in the limit  $\beta^2 \rightarrow 0$ ,  $a(\alpha_i + \alpha_j) \rightarrow -b_{ij}$ , one can rewrite this as follows (leaving only terms that can give contribution to the first order in  $\epsilon$ ):

$$(2\pi i)^{-1} [e_{\alpha_j}, e_{\alpha_i}] \\ \int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{x_m} du_2 \left( \frac{1}{u_2 - u_1 + i0} - \frac{1}{u_2 - u_1 - i0} \right) : e^{-(\alpha_i + \alpha_j, \phi(u_2))} : . \quad (43)$$

Now using the well known formula

$$\frac{1}{x + i0} - \frac{1}{x - i0} = -2i\pi \delta(x), \quad (44)$$

we obtain that (42) in the classical limit gives

$$- [e_{\alpha_j}, e_{\alpha_i}] \int_{x_{m-1}}^{x_m} du : e^{-(\alpha_i + \alpha_j, \phi(u))} : . \quad (45)$$

Next let's consider the quadratic products arising from quadratic parts of  $q$ -exponentials of fermionic roots. They look as follows:

$$\frac{-1}{(2)_{q_{\alpha_i}^{-1}}} \int_{x_{m-1}}^{x_m} du_1 W_{\alpha_i}(u_1 - i0) \int_{x_{m-1}}^{x_m} du_2 W_{\alpha_i}(u_2 + i0) e_{\alpha_i}^2. \quad (46)$$

One can rewrite this product via the ordered integrals:

$$\frac{q^{b_{ii}} - 1}{(2)_{q_{\alpha_i}^{-1}}} \int_{x_{m-1}}^{x_m} du_1 W_{\alpha_i}(u_1) \int_{x_{m-1}}^{u_1} du_2 W_{\alpha_i}(u_2) e_{\alpha_i}^2. \quad (47)$$

In the limit  $\beta^2 \rightarrow 0$  we obtain (forgetting about the terms that could give contribution of the order  $\epsilon^2$ ):

$$- \frac{ib_{ii}\beta^2}{2} \int_{x_{m-1}}^{x_m} du_1 \int_{x_{m-1}}^{u_1} du_2 (iu_1 - iu_2)^{\frac{b_{ii}\beta^2}{2}-1} e^{-2(\alpha_i, \phi(u_2))} e_{\alpha_i}^2. \quad (48)$$

Therefore the final contribution is:

$$- \int_{x_{m-1}}^{x_m} du e^{-2(\alpha_i, \phi(u))} e_{\alpha_i}^2. \quad (49)$$

Collecting now all the terms of order  $\epsilon$  we find:

$$\begin{aligned} \bar{\mathbf{L}}^{(q)}(x_m, x_{m-1}) &= 1 + \int_{x_{m-1}}^{x_m} du \left( \sum_f W_{\alpha_f}^F(u) e_{\alpha_f} + \sum_b W_{\alpha_b}^B(u) e_{\alpha_b} + \right. \\ &\quad \left. \sum_{f_1 \geq f_2} [e_{\alpha_{f_1}}, e_{\alpha_{f_2}}] W_{\alpha_{f_1} + \alpha_{f_2}}^B(u) \right) + O(\epsilon^2). \end{aligned} \quad (50)$$

Gathering the  $\bar{\mathbf{L}}^{(q)}(x_m, x_{m-1})$  it is easy to see that in the  $q \rightarrow 1$  limit  $\bar{\mathbf{L}}^{(q)}(x_m, x_{m-1})$  is equal to:

$$\begin{aligned} \bar{\mathbf{L}}^{(cl)}(2\pi, 0) &= P \exp \int_0^{2\pi} du \left( \sum_f W_{\alpha_f}^F(u) e_{\alpha_f} + \sum_b W_{\alpha_b}^B(u) e_{\alpha_b} - \right. \\ &\quad \left. \sum_{f_1 \geq f_2} [e_{\alpha_{f_1}}, e_{\alpha_{f_2}}] W_{\alpha_{f_1} + \alpha_{f_2}}^B(u) \right). \end{aligned} \quad (51)$$

Defining then  $\mathbf{L}^{(cl)} \equiv e^{\pi i p^i H^i} \bar{\mathbf{L}}^{(cl)}(2\pi, 0)$  we find that it satisfies the quadratic Poisson bracket relation (7) and defining the monodromy matrix  $\mathbf{M}^{(cl)} \equiv e^{\pi i p^i H^i} \mathbf{L}^{(cl)}$  we again obtain the classical integrability condition (9).

Now let's find the L-operator which corresponds to the monodromy matrix defined above. Let's consider the following one:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta} \Phi^i(u, \theta) H^i - \left( \sum_f e_{\alpha_f} + \sum_b \theta e_{\alpha_b} \right), \quad (52)$$

where  $D_{u,\theta} = \partial_\theta + \theta \partial_u$  is a superderivative and  $\Phi^i$  are the classical superfields with the following Poisson brackets:

$$\{D_{u,\theta} \Phi^i(u, \theta), D_{u',\theta'} \Phi^j(u', \theta')\} = \delta^{ij} D_{u,\theta} (\delta(u - u') (\theta - \theta')). \quad (53)$$

Making a gauge transformation of the above L-operator one can arrive to the fields, satisfying classical version of super W-algebras with the commutation relations provided by the Poisson brackets [13], [14].

The associated “fermionic” linear problem can be reduced to the “bosonic” one. The linear problem

$$\mathcal{L}_F \Psi(u, \theta) = (D_{u, \theta} + N_1 + \theta N_0)(\chi + \theta \eta), \quad (54)$$

where  $\Psi(u, \theta) = \chi + \theta \eta$ ,  $N_1 = \frac{i}{\sqrt{2}} \xi^i(u) H^i - \sum_f e_{\alpha_f}$ ,  $N_0 = -\partial_u \phi^i(u) H^i - \sum_b e_{\alpha_b}$ , can be reduced to the linear problem on  $\chi$ :

$$\mathcal{L}_B \chi(u) = (\partial_u + N_1^2 + N_0) \chi(u). \quad (55)$$

That is:

$$\mathcal{L}_B = \partial_u - \partial_u \phi^i(u) H^i + \left( \frac{i}{\sqrt{2}} \xi^i(u) H^i - \sum_f e_{\alpha_f} \right)^2 - \sum_b e_{\alpha_b}. \quad (56)$$

One can easily see that the monodromy matrix for the corresponding linear problem is that described above.

## 4 Integrals of Motion and Supersymmetry Invariance

It is well known that (both classical and quantum) integrability conditions lead to the involutive family of (both local and nonlocal) integrals of motion (IM). For super- versions of these systems it is also known that sometimes it is possible to include supersymmetry generator

$$G_0 = \beta^{-2} \sqrt{2} i^{-1/2} \int_0^{2\pi} du \phi''(u) \xi^l(u) = \sum_{l=0}^r \sum_{n \in \mathbb{Z}} \beta^{-2} \xi_n^l d_{-n}^l \quad (57)$$

in these series [24]. Here we will show that the transfer matrix  $\mathbf{t}^{(q)}(\lambda) = \text{str} \mathbf{M}^{(q)}(\lambda)$  commute with  $G_0$  if the simple root system is purely fermionic, that is:

$$\mathbf{t}^{(q)}(\lambda) = \text{str}(\pi(\lambda)(e^{2i\pi P^k H^k} \text{Pexp} \int_0^{2\pi} du (\sum_{f=0}^r W_{\alpha_f}^F(u) e_{\alpha_f}))), \quad (58)$$

where  $\pi$  denote some representation of the corresponding superalgebra in which the supertrace is taken.

We note the crucial property:

$$[G_0, W_\alpha^F(u)] = -\partial_u W_\alpha^B(u) \quad (59)$$

Integrating over  $u$  and multiplying by the appropriate coefficient one obtains:

$$[G_0, e_{-\alpha_i}] = \frac{W_{\alpha_i}^F(0) - W_{\alpha_i}^F(2\pi)}{q - q^{-1}}, \quad (60)$$

where  $e_{-\alpha_i}$  is represented by the vertex operator (see previous Section). Next, we will use the important Proposition (Prop. 1, Sec. 3.1 of [5]): For the objects  $A_i, B_i, \mathbf{I}$ , satisfying the commutation relations

$$[\mathbf{I}, e_{-\alpha_i}] = \frac{A_i - B_i}{q - q^{-1}}, \quad A_i e_{-\alpha_j} = q^{-b_{ij}} e_{-\alpha_j} A_i, \quad B_i e_{-\alpha_j} = q^{b_{ij}} e_{-\alpha_j} B_i, \quad (61)$$

the following relation holds:

$$[1 \otimes \mathbf{I}, \bar{R}] = \bar{R}(\sum_i e_{\alpha_i} \otimes A_i) - (\sum_i e_{\alpha_i} \otimes B_i) \bar{R}. \quad (62)$$

Applying this relations to our case we find (identifying  $A_i$  with  $W_{\alpha_i}^F(0)$ ,  $B_i$  with  $W_{\alpha_i}^F(2\pi)$  and  $\mathbf{I}$  with  $G_0$ ):

$$[G_0, \bar{\mathbf{L}}^{(q)}(2\pi, 0)] = \bar{\mathbf{L}}^{(q)}(2\pi, 0) W^B(0) - W^B(2\pi) \bar{\mathbf{L}}^{(q)}(2\pi, 0), \quad (63)$$

where

$$W^B(u) = \sum_{i=0}^r W_{\alpha_i}^B(u) e_{\alpha_i}. \quad (64)$$

Now using the property (35), periodicity properties of vertex operators:

$$W_\alpha^s(u + 2\pi) = q^{-(\alpha, \alpha)} e^{-2i\pi(\alpha, P)} W_\alpha^s(u + 2\pi) \quad (65)$$

(here  $s = B, F$ ) and cyclic property of the supertrace one obtains, multiplying both sides of (63) by  $e^{2i\pi P^k H^k}$  and taking the supertrace:

$$[G_0, \mathbf{t}^{(q)}] = 0. \quad (66)$$

We note here that if there were bosonic simple roots in the construction of the transfer-matrix the above reasonings are no longer valid, because the corresponding bosonic vertex operators commuting with  $G_0$  give the fermionic vertex operators associated with the same root vector (the same happens when we construct the superstring vertex operators [29]), but not the total derivative as in (59). It was already shown explicitly on the concrete examples that the hierarchies, based on the partly bosonic simple root systems are not invariant under the supersymmetry transformation (see e.g. [10], [11], [24]).

The affine superalgebras which allow such root systems are of the following type [12]:  $A(m, m)^{(1)} = sl(m+1, m+1)^{(1)}$ ,  $A(2m, 2m)^{(4)} = sl(2m+1, 2m+1)^{(4)}$ ,  $A(2m+1, 2m+1)^{(2)} = sl(2m+2, 2m+2)^{(2)}$ ,  $A(2m+1, 2m)^{(2)} = sl(2m+2, 2m+1)^{(2)}$ ,  $B(m, m)^{(1)} = osp(2m+1, 2m)^{(1)}$ ,  $D(m+1, m)^{(1)} = osp(2m+2, 2m)^{(1)}$ ,  $D(m, m)^{(2)} = osp(2m, 2m)^{(2)}$ ,  $D(2, 1, \alpha)^{(1)}$ .

The involutive family of the (both classical and quantum) IM in the Toda field theories have the property, that the commutators of IM with the corresponding vertex operators reduce to the total derivatives [2]:

$$[I_l, W_{\alpha_k}(u)] = \partial_u(: O_{\alpha_k}^{(l)}(u) W_{\alpha_k}(u) :) = \partial_u \Theta_k^{(l)}(u), \quad (67)$$

where  $O_{\pm}^{(k)}(u)$  is the polynomial of  $\partial_u \phi^i(u)$ ,  $\xi^i(u)$  and their derivatives. In [5] it was shown in the bosonic case (the proof is similar to the arguments above) that  $I_l$  commute with the transfer matrix, the generalization of these arguments to the super-case is straightforward.

In the next two sections we will give two examples of the KdV hierarchies related to affine superalgebra  $B(0, 1)^{(1)} \equiv osp(1|2)^{(1)}$  (super-KdV) and twisted affine superalgebra  $D(1, 1)^{(2)} \simeq C(2)^{(2)} \equiv sl(2|1)^{(2)} \simeq osp(2|2)^{(2)}$  (SUSY N=1 KdV).

## 5 Example 1: super-KdV hierarchy

The super-KdV model [30], [31] is based on the following L-operator:

$$D_{u,\theta} + D_{u,\theta} \Phi(u, \theta) h_{\alpha_0} - (e_{\alpha_1} + \theta e_{\alpha_0}), \quad (68)$$

where  $h_{\alpha_0}$ ,  $e_{\alpha_1}$ ,  $e_{\alpha_0}$  are the Chevalley generators of the upper Borel subalgebra of the  $osp(1|2)^{(1)}$  which are taken in the evaluation representation that is  $h_{\alpha_0} = -h$ ,  $e_{\alpha_1} = iv_+$ ,  $e_{\alpha_0} = \lambda X_-$ , where  $X_{\pm}$ ,  $v_{\pm}$  and  $h$  are the generators of  $osp(1|2)$  superalgebra with the following commutation relations:

$$\begin{aligned} [h, X_{\pm}] &= \pm 2X_{\pm}, & [h, v_{\pm}] &= \pm v_{\pm}, & [X_+, X_-] &= h, \\ [v_{\pm}, v_{\pm}] &= \pm 2X_{\pm}, & [v_+, v_-] &= -h, & [X_{\pm}, v_{\mp}] &= v_{\pm}, & [X_{\pm}, v_{\pm}] &= 0. \end{aligned} \quad (69)$$

Here  $p(v_{\pm}) = 1$ ,  $p(X_{\pm}) = 0$ ,  $p(h) = 0$ . The classical monodromy matrix is:

$$\mathbf{M}(\lambda) = e^{-2\pi i p h \alpha_0} P \exp \int_0^{2\pi} du \left( \frac{i}{\sqrt{2}} \xi(u) e^{-\phi(u)} e_{\alpha_1} - e^{-2\phi(u)} e_{\alpha_1}^2 + e^{2\phi(u)} e_{\alpha_0} \right). \quad (70)$$

The involutive family of the integrals of motion which can be extracted from this monodromy matrix (more precisely they arise as coefficients in the expansion in  $\lambda^{-1}$  series of the trace of the logarithm of  $\mathbf{M}$ -matrix) can be expressed via the following fields:

$$U(u) = -\phi''(u) - \phi'^2(u) - \frac{1}{2} \xi(u) \xi'(u), \quad \alpha(u) = \xi'(u) + \xi(u) \phi'(u) \quad (71)$$

generating the classical limit of the superconformal algebra under the Poisson brackets:

$$\begin{aligned} \{U(u), U(v)\} &= \delta'''(u-v) + 2U'(u)\delta(u-v) + 4U(u)\delta'(u-v), \\ \{U(u), \alpha(v)\} &= 3\alpha(u)\delta'(u-v) + \alpha'(u)\delta(u-v), \\ \{\alpha(u), \alpha(v)\} &= 2\delta''(u-v) + 2U(u)\delta(u-v). \end{aligned} \quad (72)$$

The integrals of motion are:

$$\begin{aligned} I_1^{(cl)} &= \int U(u) du, \\ I_3^{(cl)} &= \int \left( U^2(u)/2 + \alpha(u)\alpha'(u) \right) du, \\ I_5^{(cl)} &= \int \left( (U')^2(u) - 2U^3(u) + 8\alpha'(u)\alpha''(u) + 12\alpha'(u)\alpha(u)U(u) \right) du, \\ &\vdots \end{aligned} \quad (73)$$

The second one  $I_3^{(cl)}$  gives the super-KdV equation:

$$U_t = -U_{uuu} - 6UU_u - 6\alpha\alpha_{uu}, \quad \alpha_t = -4\alpha_{uuu} - 6U\alpha_u - 3U_u\alpha. \quad (74)$$

However, the supersymmetry operator  $G_0 = \int_0^{2\pi} du \alpha(u)$  can not be included in the pairwise commuting IM that can be easily seen from the second equation of (74) (i.e.  $I_3^{(cl)}$  does not commute with  $G_0$ ). Moving to the quantum case we find that the quantum analogue of the monodromy matrix is:

$$\mathbf{M}^{(\mathbf{q})}(\lambda) = e^{-2\pi i P h \alpha_0} P \exp \int_0^{2\pi} du \left( \frac{i}{\sqrt{2}} \xi(u) : e^{-\phi(u)} : e_{\alpha_1} + : e^{2\phi(u)} : e_{\alpha_0} \right), \quad (75)$$

where  $h_{\alpha_0}, e_{\alpha_0}, e_{\alpha_1}$  are now the Chevalley generators of the  $osp_q(1|2)^{(1)}$ . We did not put the letter  $q$  over the P-exponential because this is the case when for values of  $\beta^2$  from the interval  $(0,2)$  one can write the above object as a “real” P-exponential (represented via ordered integrals). Due to the presence of bosonic root  $e_{\alpha_0}$  the trace of the quantum monodromy matrix is not invariant under the supersymmetry transformation as it was in the classical case.

## 6 Example 2: SUSY N=1 KdV hierarchy

The L-operator corresponding to the SUSY N=1 KdV model [32], [33] is the following one:

$$\mathcal{L}_F = D_{u,\theta} - D_{u,\theta}\Phi(u, \theta)h_{\alpha_1} - (e_{\alpha_0} + e_{\alpha_1}), \quad (76)$$

where  $h_{\alpha_1}, e_{\alpha_0}, e_{\alpha_1}$  are the Chevalley generators of the twisted affine Lie superalgebra  $C(2)^{(2)}$  with such set of commutation relations:

$$\begin{aligned} [h_{\alpha_1}, h_{\alpha_0}] &= 0, \quad [h_{\alpha_0}, e_{\pm\alpha_1}] = \mp e_{\pm\alpha_1}, \quad [h_{\alpha_1}, e_{\pm\alpha_0}] = \mp e_{\pm\alpha_0}, \\ [h_{\alpha_i}, e_{\pm\alpha_i}] &= \pm e_{\pm\alpha_i}, \quad [e_{\pm\alpha_i}, e_{\mp\alpha_j}] = \delta_{i,j}h_{\alpha_i}, \quad (i, j = 0, 1), \\ ad_{e_{\pm\alpha_0}}^3 e_{\pm\alpha_1} &= 0, \quad ad_{e_{\pm\alpha_1}}^3 e_{\pm\alpha_0} = 0 \end{aligned} \quad (77)$$

Here  $p(h_{\alpha_{0,1}}) = 0$ ,  $p(e_{\pm\alpha_{0,1}}) = 1$ , i.e. both simple roots are fermionic. The classical monodromy matrix is:

$$\begin{aligned} \mathbf{M} &= e^{2\pi i p h_{\alpha_1}} P \exp \int_0^{2\pi} du \left( \frac{i}{\sqrt{2}} \xi(u) e^{-\phi(u)} e_{\alpha_1} \right. \\ &\quad \left. - \frac{i}{\sqrt{2}} \xi(u) e^{\phi(u)} e_{\alpha_0} - e_{\alpha_1}^2 e^{-2\phi(u)} - e_{\alpha_0}^2 e^{2\phi(u)} - [e_{\alpha_1}, e_{\alpha_0}] \right). \end{aligned} \quad (78)$$

The series of the integrals of motion starts with the following ones:

$$\begin{aligned} I_1^{(cl)} &= \frac{1}{2\pi} \int U(u) du, \\ I_3^{(cl)} &= \frac{1}{2\pi} \int \left( U^2(u) + \alpha(u)\alpha'(u)/2 \right) du, \\ I_5^{(cl)} &= \frac{1}{2\pi} \int \left( U^3(u) - (U')^2(u)/2 - \alpha'(u)\alpha''(u)/4 - \alpha'(u)\alpha(u)U(u) \right) du, \\ &\vdots \end{aligned} \quad (79)$$



The fields  $U$  and  $\alpha$  are defined in terms of the free fields as in previous section, but now one can unify them into one superfield:

$$\mathcal{U}(u, \theta) \equiv D_{u, \theta} \Phi(u, \theta) \partial_u \Phi(u, \theta) - D_{u, \theta}^3 \Phi(u, \theta) = -\theta U(u) - i\alpha(u)/\sqrt{2}. \quad (80)$$

The second IM from (79) generates the first nontrivial evolution equation, the SUSY  $N=1$  KdV:

$$\mathcal{U}_t = -\mathcal{U}_{uuu} + 3(\mathcal{U} D_{u, \theta} \mathcal{U})_u, \quad (81)$$

or in components:

$$U_t = -U_{uuu} - 6UU_u - \frac{3}{2}\alpha\alpha_{uu}, \quad \alpha_t = -4\alpha_{uuu} - 3(U\alpha)_u. \quad (82)$$

Now one can see that unlike the previous example the supersymmetry generator  $\int_0^{2\pi} du \alpha(u)$  commutes with  $I_3^{(cl)}$  and can be included in the involutive family of IM [24]. Moreover, the results of the Section 2 yield that the quantum monodromy matrix have the following form:

$$\mathbf{M} = e^{2\pi i Ph_{\alpha_1}} \text{Pexp}^{(q)} \int_0^{2\pi} du (W_-(u) e_{\alpha_1} + W_+(u) e_{\alpha_0}), \quad (83)$$

where  $W_{\pm} = \int d\theta : e^{\pm\Phi(u, \theta)} :$ . Due to the fact that both roots are fermionic the supersymmetry generator commutes with the transfer matrix.

This SUSY  $N=1$  KdV model was studied from a point of view of the Quantum Inverse Scattering Method in [22]. There were constructed the analogues of Baxter's Q-operator, providing the following functional relations with the transfer-matrices:

$$\begin{aligned} \mathbf{t}_{\frac{1}{4}}(\lambda) \mathbf{Q}_{\pm}(\lambda) &= \pm \mathbf{Q}_{\pm}(q^{\frac{1}{2}}\lambda) \mp \mathbf{Q}_{\pm}(q^{-\frac{1}{2}}\lambda), \\ \mathbf{t}_{\frac{1}{2}}(q^{\frac{1}{4}}\lambda) \mathbf{Q}_{\pm}(\lambda) &= \mp \mathbf{t}_{\frac{1}{4}}(q^{\frac{1}{2}}\lambda) \mathbf{Q}_{\pm}(q^{-\frac{1}{2}}\lambda) + \mathbf{Q}_{\pm}(q\lambda), \end{aligned} \quad (84)$$

and the fusion relations between transfer-matrices in different representations:

$$\mathbf{t}_j(q^{\frac{1}{4}}\lambda) \mathbf{t}_j(q^{-\frac{1}{4}}\lambda) = \mathbf{t}_{j+\frac{1}{4}}(\lambda) \mathbf{t}_{j-\frac{1}{4}}(\lambda) + (-1)^{4j}, \quad (85)$$

which for the case when  $q$  is a root of unity can be transformed into the Thermodynamic Bethe Ansatz equations of  $D_{2N}$  type [35].

It should be noted also that the associated Toda field theory is a well known  $N=1$  SUSY sinh-Gordon model [33], [34] with the action:

$$\frac{1}{\beta^2} \int d^2u d^2\theta (D_{u,\theta} \Phi \bar{D}_{\bar{u},\bar{\theta}} \Phi + m^2 \cosh(\Phi)) \quad (86)$$

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## 7 Appendix

The affine Lie superalgebra in has the following commutation relations between its Chevalley generators ( $H^i$  forms a basis in the Cartan subalgebra of the underlying simple Lie superalgebra and  $e_{\pm\alpha_k}$  are the generators associated with positive and negative simple roots of the whole affine algebra) [27]:

$$\begin{aligned} [H^i, e_{\alpha_k}] &= \alpha_k^i e_{\alpha_k}, \quad [e_{\alpha_k}, e_{-\alpha_l}] = \delta_{kl} [h_{\alpha_k}]_q, \quad ad_{e_{\pm\alpha_k}}^{1-a_{kj}} e_{\pm\alpha_j} = 0, \\ [[e_{\pm\alpha_r}, e_{\pm\alpha_s}]_q, [e_{\pm\alpha_r}, e_{\pm\alpha_p}]_q]_q &= 0 \end{aligned} \quad (87)$$

if  $(\alpha_r, \alpha_r) = (\alpha_s, \alpha_p) = (\alpha_r, \alpha_s + \alpha_p) = 0$ , in this case it is usually said that  $\alpha_r$  is a “grey” root, which is between two roots  $\alpha_s, \alpha_p$  on the Dynkin diagram [25], [26]. The definition of the super q-commutator is:

$$ad_{e_\alpha}^{(q)} e_\beta = e_\alpha e_\beta - (-1)^{p(\alpha)p(\beta)} q^{(\alpha,\beta)} e_\beta e_\alpha, \quad (88)$$

where  $p(\alpha)$  is equal to 1 when  $\alpha$  is a fermionic root, and to 0 if  $\alpha$  is a bosonic root. The universal R-matrix for the contragredient Lie superalgebra of finite growth (affine algebra as a particular case) has the following structure:

$$\mathbf{R} = K \bar{R} = K \left( \prod_{\alpha \in \Delta_+}^{\rightarrow} R_\alpha \right), \quad (89)$$

where  $\bar{R}$  is a reduced R-matrix and  $R_\alpha$  are defined by the formulae:

$$R_\alpha = exp_{q_\alpha^{-1}}((-1)^{p(\alpha)}(q - q^{-1})(a(\alpha))^{-1}(e_\alpha \otimes e_{-\alpha})) \quad (90)$$

for real roots and

$$R_{n\delta} = \exp((-1)^{p(n\delta)}(q - q^{-1})\left(\sum_{i,j}^{mult} c_{ij}(n)e_{n\delta}^{(i)} \otimes e_{-n\delta}^{(j)}\right)) \quad (91)$$

for pure imaginary roots. Here  $\Delta_+$  is the reduced positive root system (the bosonic roots which are two times fermionic roots are excluded). The generators corresponding to the composite roots are defined according to the construction of the Cartan-Weyl basis given in [27]. For example the generators of the type  $e_{\pm\alpha_{f_1} \pm \alpha_{f_2}}$  are constructed by means of the following q-commutators:

$$e_{\alpha_{f_1} + \alpha_{f_2}} = [e_{\alpha_{f_2}}, e_{\alpha_{f_1}}]_{q^{-1}}, \quad e_{-\alpha_{f_1} - \alpha_{f_2}} = [e_{-\alpha_{f_1}}, e_{-\alpha_{f_2}}]_q. \quad (92)$$

The  $a(\alpha)$  coefficients are defined as follows:

$$[e_\gamma, e_{-\gamma}] = a(\gamma) \frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}}. \quad (93)$$

We will need the values of  $a(\gamma)$  when  $\gamma$  is equal to  $\alpha_{f_1} + \alpha_{f_2}$ , where  $\alpha_{f_1}$  and  $\alpha_{f_2}$  are fermionic simple roots:

$$a(\alpha_{f_1} + \alpha_{f_2}) = \frac{q^{-b_{f_1 f_2}} - q^{b_{f_1 f_2}}}{q - q^{-1}}. \quad (94)$$

The q-exponentials in (89) are defined in the usual way:

$$\exp_q(x) = 1 + x + \frac{x^2}{(2)_q!} + \dots + \frac{x^n}{(n)_q!} + \dots = \sum_{n \geq 0} \frac{x^n}{(n)_q!} \quad (95)$$

$$(a)_q \equiv \frac{q^a - 1}{q - 1}, \quad q_\alpha \equiv (-1)^{p(\alpha)} q^{(\alpha, \alpha)}.$$

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